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ON THE DECOUPLING OF CIRCULATORY GYROSCOPIC SYSTEMS BY CONGRUENCE

Abstract: The possibility of decomposition of linear n-degree-of-freedom systems with potential, circulatory and gyroscopic forces is studied. Criteria that contain conditions for the existence of a real linear coordinate transformation, leading to the separation of such a system into independent subsystems of dimensions no more than two, are proved. The conditions are expressed in terms of the coefficient matrices of the original system. Several specific results are also obtained as consequences of the criteria. Three numerical examples are supplied to illustrate the applicability and correctness of the obtained results.

Keywords: linear system; potential forces; circulatory forces; gyroscopic forces; decoupling; congruence transformation

1. INTRODUCTION

A remarkable class of linear dynamical systems is associated with potential (conservative), circulatory and gyroscopic forces and can be described by

$$\tilde{M}\ddot{q} + \tilde{G}\dot{q} + \tilde{N}q + \tilde{K}q = 0\,, \tag{1}$$

where $\tilde{M}$, $\tilde{G}$, $\tilde{N}$ and $\tilde{K}$ are n by n constant real matrices; the inertia matrix $\tilde{M}$ is symmetric and positive definite ($\tilde{M} = \tilde{M}^T > 0$), $\tilde{G}$ and $\tilde{N}$ are skew-symmetric ($\tilde{G} = -\tilde{G}^T$, $\tilde{N} = -\tilde{N}^T$) and $\tilde{K}$ is symmetric. The n-vector of generalized coordinates is denoted by q, and the dots indicate

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differentiation with respect to the time t. The terms involving $\tilde{G}$, $\tilde{N}$ and $\tilde{K}$ correspond respectively to gyroscopic, circulatory and potential forces. Gyroscopic forces occur in, for instance, the rotary motion in rotating flexible machinery, spinning elastic systems, astrodynamics, satellite control, problems related to the motion of charged systems in magnetic fields, order reduction of systems with symmetries, and when using rotating frames of reference in analytical dynamics. While it is easy to visualize the presence of potential and gyroscopic forces, the presence of circulatory forces is perhaps some-what less intuitive. Yet, such forces arise in many areas of real-life applications. Some examples are control of two legged walking robots, self-oscillations (shimmy) in aircraft wheels, flutter in flexible structures, dynamics of brake squealing, and wear in paper calendars [1].

Equation (1) represents a set of coupled second-order ordinary-differential equations and can be obtained by the application of Lagrange's equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} + \frac{\partial \Phi}{\partial \dot{q}} = 0 \tag{2}$$

with the Lagrangian

$$L(q,\dot{q}) = \frac{1}{2}\dot{q}^T \tilde{M}\dot{q} + \frac{1}{2}\dot{q}^T \tilde{G}q - \frac{1}{2}q^T \tilde{K}q \tag{3}$$

and the "dissipative" function

$$\Phi(q,\dot{q}) = \dot{q}^T \tilde{N}q \, . \tag{4}$$

Consider a change of coordinates from q to p defined by the real linear transformation

$$q = Pp \Leftrightarrow p = P^{-1}q \tag{5}$$

where P can be any nonsingular real matrix. Noting Eqs. (3) and (4), this transformation of coordinates causes $\tilde{M}$, $\tilde{K}$, $\tilde{N}$ and $\tilde{G}$ to be congruently transformed, i. e.,

$$\tilde{M} \to M = P^T \tilde{M}P \, , \; \tilde{K} \to K = P^T \tilde{K}P \, , \; \tilde{N} \to N = P^T \tilde{N}P \, , \; \tilde{G} \to G = P^T \tilde{G}P \, . \tag{6}$$

If $\tilde{N} = 0$ and $\tilde{G} = 0$ (pure potential system or conservative non-gyroscopic system), one can always find the transformation matrix P so that the new inertia and potential (stiffness) matrices are diagonal, i.e., in new coordinates p, called normal (principal or modal) coordinates, the system is decomposed into n independent single-degree-of-freedom subsystems. This classical result was established by Weierstrass in 1858 in the context of simultaneous reduction of two quadratic forms to sums of squares (see [2]). The procedure for decoupling such systems is well known and is called modal analysis. When $\tilde{N} \neq 0$ and/or

$\tilde{G} \neq 0$, the system is not completely decomposable because changes of coordinates (5) that makes M and K diagonal retains N and/or G as skew-symmetric matrices. However, sometimes it might be decomposed into several independent subsystems. Also, it is worth pointing out that the minimum number of degrees of freedom necessary to incorporate circulatory and/or gyroscopic effects is two. Therefore, it is natural to ask whether or not we can decompose system (1) into independent subsystems, each of which has degree-of-freedom no more than two, by means of a change of coordinates (i.e. using a congruence transformation). The intent of this paper is to show that multi-degree of freedom circulatory gyroscopic potential systems can be uncoupled when certain conditions are satisfied.

The next short section presents some algebraic results, less known to a wider audience of readers, which are basic to our further development. In Section 3, the results are formulated, proved and discussed.

2. MATHEMATICAL BACKGROUND

The following assertion gives some useful properties of real skew-symmetric matrices (see [3,4], for example).

Lemma 1. *Let B be an $n \times n$ real skew-symmetric matrix. Then:*

(a) $r = rankB$ is even.

(b) B has single zero eigenvalue of multiplicity $n - r$, and r pure imaginary eigenvalues in pairs $\pm i\beta_j$, $i = \sqrt{-1}$, $j = 1,...,r/2$, which are all simple or semi-simple.

(c) There exists a real orthogonal matrix U such that

$$U^T BU = diag\left(\beta_1 J_2,...,\beta_{r/2} J_2,0,...,0\right), \tag{7}$$

where J_2 is the symplectic unit two-dimensional matrix, i.e,

$$J_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The block-diagonal (quasi-diagonal) matrix (7) is the canonical (simplest possible) form of a skew-symmetric matrix with respect to orthogonal similarities, while the canonical form for a real symmetric matrix is, of course, a diagonal matrix consisting of its eigenvalues along the diagonal. Also, notice that form (7) is recognized as the real Jordan form for B and hence plays a fundamental algebraic role [4].

The following assertion plays a key role in our further considerations. It is a counterpart of the well-known result which states that two real symmetric matrices can be simultaneously diagonalized by a real orthogonal transformation if and only if they commute in multiplication [3].

Lemma 2.[5] *Let* $A = A^T$ *and* $B = -B^T$ *be* $n \times n$ *real matrices, and let be* $r = rankB$. *Necessary and sufficient conditions that there exists a real orthogonal matrix* U *such that*

$$U^T AU = \hat{A} = diag(\alpha_1,...,\alpha_n) \tag{8}$$

and

$$U^T BU = \hat{B} = diag\left(\beta_1 J_2,...,\beta_{r/2} J_2,0,...,0\right), \tag{9}$$

are that

$$AB^2 = B^2 A \tag{10}$$

and

$$(AB)^2 = (BA)^2. \tag{11}$$

It is clear that conditions (10) and (11) imply the symmetry of the matrices AB^2 and $(AB)^2$, and vice versa.

Remark 1. Since $\hat{A}$ is diagonal and $\hat{B}$ quasi-diagonal, it is easy to see that the matrices $\hat{B}^2$ and $\hat{B}\hat{A}\hat{B}$ are diagonal, i. e.,

$$\hat{B}^2 = -diag(\beta_1^2 I_2,...,\beta_{r/2}^2 I_2,0,...,0) \tag{12}$$

and

$$\hat{B}\hat{A}\hat{B} = -diag(\beta_1^2\alpha_2,\beta_1^2\alpha_1,...,\beta_{r/2}^2\alpha_r,\beta_{r/2}^2\alpha_{r-1},0,...,0). \tag{13}$$

Lemma 3. *If all nonzero eigenvalues of the skew-symmetric matrix* B *are distinct, then condition (10) implies condition (11).*

Proof. **A**ccording to Lemma 1, there exists a real orthogonal matrix U such that

$$B = U\begin{bmatrix} \hat{B} & 0 \\ 0 & 0_{n-r} \end{bmatrix}U^T, \tag{14}$$

and

$$A = U\begin{bmatrix} \hat{A} & \overline{A} \\ \overline{A}^T & \hat{\hat{A}} \end{bmatrix} U^T, \tag{15}$$

where

$$\hat{B} = diag\left(\beta_1 J_2, ..., \beta_{r/2} J_2\right), \tag{16}$$

and 0_{n-r} is (n-r)-dimensional zero matrix, $\hat{A}$ and $\hat{\hat{A}}$ are r and (n-r) dimensional symmetric matrices respectively, and $\overline{A}$ is r by (n-r) matrix. Then, condition (10) yields $\overline{A} = 0$ because $\hat{B}$ is nonsingular, and

$$\hat{A}\hat{B}^2 = \hat{B}^2 \hat{A} \tag{17}$$

It follows from (16) that $\hat{B}^2 = -diag(\beta_1^2 I_2, ..., \beta_{r/2}^2 I_2)$, where I_2 denotes 2 by 2 identity matrix. Next, after partitioning the symmetric matrix $\hat{A}$, as $\hat{A} = \left[\hat{A}_{ij}\right]_{i,j=1}^{r/2}$ with two-dimensional sub-matrices $\hat{A}_{ij}$, condition (17) becomes

$$\left[\beta_j^2 \hat{A}_{ij}\right]_{i,j=1}^{r/2} = \left[\beta_i^2 \hat{A}_{ij}\right]_{i,j=1}^{r/2},$$

or $(\beta_i^2 - \beta_j^2)\hat{A}_{ij} = 0$, which yields $\hat{A}_{ij} = 0$ for $i \neq j$ since, in view of the assumption, all numbers β_1, …, $\beta_{r/2}$ are distinct. Thus, the matrix A that satisfy condition (10) must be of the form

$$A = U\begin{bmatrix} diag(\hat{A}_{ii})_{i=1}^{r/2} & 0 \\ 0 & \hat{\hat{A}} \end{bmatrix} U^T. \tag{18}$$

where $\hat{A}_{ii}$, $i = 1, ..., r/2$, are two by two symmetric matrices and $\hat{\hat{A}}$ is an (n-r)-dimensional symmetric matrix. Now, it follows that $(AB)^2 = (BA)^2$, where the matrices B and A are determined by Eqs. (14) and (18), since as the easy confirm that the two-dimensional sub-matrices $\hat{A}_{ii}$ and $\beta_i J_2$ satisfy this condition. □

3. RESULTS

We begin with the observation that in principal coordinates a two-degree-of-freedom system (1) has the form

$$\ddot{p} + gJ_2 \dot{p} + \nu J_2 p + \Lambda p = 0. \tag{19}$$

Here p is two-dimensional real vector of principal coordinates, $\Lambda = diag(\lambda_1, \lambda_2)$ and λ_1, λ_2, ν and g are real numbers.

Suppose that $rank\tilde{N} = 2m$ and that a transformation (5) decomposes the system (1) into independent subsystems, each of which has degree-of-freedom no more than two. In this case, in view of the above observation, we can assume that the transformed system has the following form

$$\ddot{p} + \Gamma\dot{p} + \mathrm{N}p + \Lambda p = 0 \tag{20}$$

with

$$\Lambda = diag(\lambda_1, \lambda_2, ..., \lambda_n), \tag{21}$$

$$\mathrm{N} = diag(\nu_1 J_2, ..., \nu_m J_2, 0, ..., 0), \tag{22}$$

and

$$\Gamma = diag\left(g_1 J_2, ..., g_{[n/2]} J_2, 0\right) \tag{23}$$

where $\lambda_j, g_j \in \Re$, $0 \neq \nu_j \in \Re$ and $[n/2]$ denotes the integer of $n/2$. It should be noted that when n is even then there is no scalar zero element at the end of the main diagonal of the matrix Γ, while in the case of odd n it always appears.

Theorem 1. *Let* $\tilde{M} = \tilde{M}^T > 0$, $\tilde{K} = \tilde{K}^T$, $\tilde{N} = -\tilde{N}^T$, $\tilde{G} = -\tilde{G}^T$ *and* $rank\tilde{N} = 2m$. *If* $\tilde{G} = \gamma\tilde{N}$ *where* $\gamma \in \Re$, *necessary and sufficient conditions for Eq. (1) to be transformed to Eq. (20) with*

$$\Lambda = diag(\lambda_1, \lambda_2, ..., \lambda_n),\ \mathrm{N} = diag(\nu_1 J_2, ..., \nu_m J_2, 0, ..., 0),\ \Gamma = \gamma\mathrm{N} \tag{24}$$

using a real linear change of coordinates are

$$\tilde{K}\tilde{M}^{-1}\tilde{N}\tilde{M}^{-1}\tilde{N} = \tilde{N}\tilde{M}^{-1}\tilde{N}\tilde{M}^{-1}\tilde{K} \tag{25}$$

and

$$(\tilde{K}\tilde{M}^{-1}\tilde{N}\tilde{M}^{-1})^2 = (\tilde{N}\tilde{M}^{-1}\tilde{K}\tilde{M}^{-1})^2. \tag{26}$$

Proof. *Necessity*. Let nonsingular matrix P be such that $P^T\tilde{M}P = I$, $P^T\tilde{N}P = \mathrm{N}$, and $P^T\tilde{K}P = \Lambda$ with N and Λ quasi-diagonal skew symmetric and diagonal matrices as in (24). Then $P^T\tilde{G}P = \gamma P^T\tilde{N}P = \gamma\mathrm{N} = \Gamma$. On the other hand we have $\tilde{M}^{-1} = PP^T$, $\tilde{N} = P^{-T}\mathrm{N}P^{-1}$, and $\tilde{K} = P^{-T}\Lambda P^{-1}$, so $\tilde{K}\tilde{M}^{-1}\tilde{N}\tilde{M}^{-1}\tilde{N} =$ $P^{-T}\Lambda\mathrm{N}^2P^{-1}$ $= P^{-T}\mathrm{N}^2\Lambda P^{-1} = \tilde{N}\tilde{M}^{-1}\tilde{N}\tilde{M}^{-1}\tilde{K}$ since, in view of Remark 1, N^2 is diagonal and therefore it commutes with the diagonal matrix Λ. Also, $(\tilde{K}\tilde{M}^{-1}\tilde{N}\tilde{M}^{-1})^2 = (P^{-T}\Lambda \mathrm{N} P^T)^2 = P^{-T}\Lambda\mathrm{N}\Lambda\mathrm{N}P^T = P^{-T}\mathrm{N}\Lambda\mathrm{N}\Lambda P^T = (\tilde{N}\tilde{M}^{-1}\tilde{K}\tilde{M}^{-1})^2$ since, again by Remark 1, the matrix $\mathrm{N}\Lambda\mathrm{N}$ is diagonal and, therefore, commutes with Λ. Thus, if there is a real congruence transformation $q = Pp$ which transforms Eq. (1) to the form (20), (24) then conditions (25) and (26) are satisfied.

Sufficiency. Suppose that conditions (25) and (26) are satisfied. Making the transformation $x = \tilde{M}^{1/2}q$, where the exponent ½ indicates the unique positive definite square root of the matrix $\tilde{M}$, from (6) we get $M = \tilde{M}^{-1/2}\tilde{M}\tilde{M}^{-1/2} = I$, $K = K^T = \tilde{M}^{-1/2}\tilde{K}\tilde{M}^{-1/2}$, $N = -N^T = \tilde{M}^{-1/2}\tilde{N}\tilde{M}^{-1/2}$ and $G = -G^T = \tilde{M}^{-1/2}\tilde{G}\tilde{M}^{-1/2} = \gamma N$. Now condition (25) after multiplication from the left and right by $\tilde{M}^{-1/2}$ becomes $KN^2 = N^2K$, while condition (26) after multiplication from the left by $\tilde{M}^{-1/2}$ and from the right by $\tilde{M}^{1/2}$ becomes $(KN)^2 = (NK)^2$. Then, by virtue of Lemma 2, there is an orthogonal matrix U that transforms simultaneously K and N to the forms

$$U^T KU = \Lambda = diag(\lambda_1,...,\lambda_n),\ U^T NU = \mathrm{N} = diag\left(\nu_1 J_2,...,\nu_m J_2,0,...,0\right)$$

and, consequently, the transformation $q = \tilde{M}^{-1/2}Up$ reduces Eq. (1) to the form (20), (24) since $U^T GU = \gamma U^T NU = \gamma\mathrm{N}$. The theorem is proved. □

Remark 2. It is not difficult to see that the real numbers λ_j (j = 1,…,n) are eigenvalues of the matrix $\tilde{M}^{-1}\tilde{K}$, while the nonzero eigenvalues of $\tilde{M}^{-1}\tilde{N}$ are $\pm i\nu_j$ (j = 1,…,m).

Note that the decoupling conditions (25) and (26) trivially hold when either $\tilde{K} = 0$ or $\tilde{N} = 0$. In the first case the system can be reduced to the form $\ddot{p} + \gamma\mathrm{N}\dot{p} + \mathrm{N}p = 0$, while in the second case, as well-known and previously mentioned in the Introduction, the system can be transformed to the completely decupled form $\ddot{p} + \Lambda p = 0$. If $\tilde{G} = 0$ ($\gamma = 0$), Theorem 1 gives necessary and sufficient conditions for the quasi-diagonalization of circulatory systems [6,7].

Remark 3. Obviously, in the so-called pseudo-normal coordinates, i. e., coordinates obtained through the transformation $q = \tilde{M}^{1/2}x$, the equations for motion of the system are given by

$$\ddot{x} + G\dot{x} + Nx + Kx = 0, \tag{27}$$

where $K = K^T = \tilde{M}^{-1/2}\tilde{K}\tilde{M}^{-1/2}$, $N = -N^T = \tilde{M}^{-1/2}\tilde{N}\tilde{M}^{-1/2}$ and $G = -G^T = \tilde{M}^{-1/2}\tilde{G}\tilde{M}^{-1/2}$, and conditions (25), (26) become

$$KN^2 = N^2K \qquad (28)$$

and

$$(KN)^2 = (NK)^2. \qquad (29)$$

Example 1. Consider the four-degree-of-freedom system described by Eq. (1) in which

$$\tilde{M} = I_4,\ \tilde{K} = K = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},\ \tilde{N} = N = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{bmatrix},\ \tilde{G} = G = \frac{1}{2}\tilde{N}, \qquad (30)$$

where I_4 is the 4 by 4 identity matrix.

The matrices K and N in (30) satisfy condition (28) because $N^2 = -I_4$. Next we calculate

$$KN = 2\begin{bmatrix} 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & 0 \\ 2 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

This matrix is skew-symmetric and then the matrix $(KN)^2$ is symmetric, i. e. condition (29) is also satisfied. Because conditions (28) and (29) of Remark 3 are satisfied and taking into account that $rankN = 4$, the system can be transformed by a real congruence transformation into two independent two-dimensional subsystems. Indeed, one easily verifies that the coordinate transformation $x = Up$, where the columns of the transformation matrix U are the following orthonormal eigenvectors of the matrix K

$$u_1 = \frac{\sqrt{2+\sqrt{2}}}{2}\begin{bmatrix}1 & 1-\sqrt{2} & 0 & 0\end{bmatrix}^T,\ u_2 = \frac{\sqrt{2+\sqrt{2}}}{2}\begin{bmatrix}0 & 0 & 1 & 1-\sqrt{2}\end{bmatrix}^T,$$

$$u_3 = \frac{\sqrt{2-\sqrt{2}}}{2}\begin{bmatrix}1 & 1+\sqrt{2} & 0 & 0\end{bmatrix}^T,\ u_4 = \frac{\sqrt{2-\sqrt{2}}}{2}\begin{bmatrix}0 & 0 & 1 & 1+\sqrt{2}\end{bmatrix}^T,$$

transforms the system into decomposed form

$$\begin{bmatrix}\ddot{p}_1\\ \ddot{p}_2\end{bmatrix}+\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}\dot{p}_1\\ \dot{p}_2\end{bmatrix}+\begin{bmatrix}0 & 2\\ -2 & 0\end{bmatrix}\begin{bmatrix}p_1\\ p_2\end{bmatrix}-(\sqrt{2}+1)\begin{bmatrix}p_1\\ p_2\end{bmatrix}=0$$

$$\begin{bmatrix}\ddot{p}_3\\ \ddot{p}_4\end{bmatrix}+\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}\dot{p}_3\\ \dot{p}_4\end{bmatrix}+\begin{bmatrix}0 & 2\\ -2 & 0\end{bmatrix}\begin{bmatrix}p_3\\ p_4\end{bmatrix}+(\sqrt{2}-1)\begin{bmatrix}p_3\\ p_4\end{bmatrix}=0\,.$$

Corollary 1. *Let* $\tilde{M}=\tilde{M}^T>0$, $\tilde{K}=\tilde{K}^T$, $\tilde{N}=-\tilde{N}^T$ *and* $rank\tilde{N}=2m$. *If* $\tilde{G}=\gamma\tilde{N}$ *with* $\gamma\in\Re$ *and all nonzero eigenvalues of the matrix* $\tilde{M}^{-1}\tilde{N}$ *are distinct, then there exists a real linear change of coordinates that transforms Eq. (1) to the form (20), (24) if and only if the following condition holds*

$$\tilde{K}\tilde{M}^{-1}\tilde{N}\tilde{M}^{-1}\tilde{N}=\tilde{N}\tilde{M}^{-1}\tilde{N}\tilde{M}^{-1}\tilde{K}\,. \tag{31}$$

Proof. The matrices $\tilde{M}^{-1}\tilde{N}$ and $N=-N^T=\tilde{M}^{-1/2}\tilde{N}\tilde{M}^{-1/2}$ have the same eigenvalues. Then, if all nonzero eigenvalues of the matrix $\tilde{M}^{-1}\tilde{N}$ are distinct, according to Remark 3 and Lemma 3, condition (25) implies condition (26). □

Remark 4. In the case of a three-degree-of-freedom system described by Eq. (1), $\tilde{G}=\gamma\tilde{N}$ together with (31) are necessary and sufficient conditions for its decomposition to the form (20), (24). It follows from the fact that in this case in Eqs. (20)-(23) the matrix Γ is proportional to the matrix N, and that any nonzero three-dimensional skew-symmetric matrix has distinct eigenvalues.

Example 2. Consider the system (1) with

$$\tilde{M}=\begin{bmatrix}5 & 0 & 4\\ 0 & 4 & 0\\ 4 & 0 & 5\end{bmatrix},\tilde{K}=\begin{bmatrix}13 & -3 & 14\\ -3 & 10 & 3\\ 14 & 3 & 13\end{bmatrix},\ \tilde{N}=\begin{bmatrix}0 & 2 & 0\\ -2 & 0 & 2\\ 0 & -2 & 0\end{bmatrix},\tilde{G}=-3\tilde{N}\,. \tag{32}$$

We calculate

$$\tilde{M}^{-1}=\frac{1}{9}\begin{bmatrix}5 & 0 & -4\\ 0 & 2.25 & 0\\ -4 & 0 & 5\end{bmatrix}$$

and

$$\tilde{K}\tilde{M}^{-1}\tilde{N}\tilde{M}^{-1}\tilde{N}=\begin{bmatrix}1 & 6 & -1\\ 6 & -20 & -6\\ -1 & -6 & 1\end{bmatrix}=(\tilde{K}\tilde{M}^{-1}\tilde{N}\tilde{M}^{-1}\tilde{N})^T=\tilde{N}\tilde{M}^{-1}\tilde{N}\tilde{M}^{-1}\tilde{K}$$

and, according to Corollary 1, there exists principal coordinates in which the system decomposes into one two-degree and one single degree of freedom subsystems. In order to obtain the transformation matrix P that decomposes the system, we look for solution of the generalized symmetric eigenvalue problem $\tilde{K}u = \lambda \tilde{M}u$. We get the eigenvalues and corresponding eigenvectors normalized with respect to the mass matrix $\tilde{M}$, as follows

$$\lambda_1 = -2,\ u_1 = \frac{1}{\sqrt{22}}\begin{bmatrix}3 & 1 & -3\end{bmatrix}^T;\ \lambda_2 = 3,\ u_2 = \frac{1}{3\sqrt{2}}\begin{bmatrix}1 & 0 & 1\end{bmatrix}^T;$$

$$\lambda_3 = \frac{7}{2},\ u_3 = \frac{1}{\sqrt{11}}\begin{bmatrix}1 & -\frac{3}{2} & -1\end{bmatrix}^T.$$

Since $\tilde{G}u_2 = 0$, we introduce principal coordinates $p = \begin{bmatrix}p_1 & p_2 & p_3\end{bmatrix}^T$ by the transformation $q = Pp$, where $P = \begin{bmatrix}u_1 & u_3 & u_2\end{bmatrix}$. Now it is easy to verify that this transformation reduces system (1), (32) to the form

$$\begin{bmatrix}\ddot{p}_1 \\ \ddot{p}_2\end{bmatrix} + 3\sqrt{2}\begin{bmatrix}0 & 1 \\ -1 & 0\end{bmatrix}\begin{bmatrix}\dot{p}_1 \\ \dot{p}_2\end{bmatrix} + \sqrt{2}\begin{bmatrix}0 & -1 \\ 1 & 0\end{bmatrix}\begin{bmatrix}p_1 \\ p_2\end{bmatrix} + \begin{bmatrix}-2p_1 \\ 3.5p_2\end{bmatrix} = 0$$

$$\ddot{p}_3 + 3p_3 = 0.$$

Remark 5. When $\tilde{G} = \gamma\tilde{N}$ and $rank\tilde{N} = 2$, condition (31) is necessary and sufficient for decomposition of the system described by Eq. (1) to one circulatory gyroscopic subsystem with two degrees of freedom and (n-2) potential subsystems with single degree of freedom.

Theorem 2. *Let* $\tilde{M} = \tilde{M}^T > 0$, $\tilde{K} = \tilde{K}^T$, $\tilde{N} = -\tilde{N}^T$, $\tilde{G} = -\tilde{G}^T$ *and* $rank\tilde{N} = 2m$. *If the following tree conditions are satisfied*

$$\tilde{K}\tilde{M}^{-1}\tilde{N} = \tilde{N}\tilde{M}^{-1}\tilde{K},\ \tilde{K}\tilde{M}^{-1}\tilde{G} = \tilde{G}\tilde{M}^{-1}\tilde{K},\ \tilde{G}\tilde{M}^{-1}\tilde{N} = \tilde{N}\tilde{M}^{-1}\tilde{G}, \tag{33}$$

then there exists a real linear change of coordinates that transforms Eq. (1) to the form (20) with

$$\Lambda = diag(\lambda_1 I_2, \ldots, \lambda_m I_2, \lambda_{2m+1}, \ldots, \lambda_n), \tag{34}$$

$$\mathrm{N} = diag(\nu_1 J_2, \ldots, \nu_m J_2, 0, \ldots, 0), \tag{35}$$

and

$$\Gamma = diag\left(g_1 J_2, \ldots, g_{[n/2]} J_2, 0\right), \tag{36}$$

where $\lambda_j, g_j \in \Re$, *and* $0 \neq \nu_j \in \Re$. *If* $m < [n/2]$, *then for any* $g_{m+k} \neq 0$**,** $\lambda_{2m+2k-1} = \lambda_{2m+2k}$, $k \in \{1,...,[n/2]-m\}$**.**

Proof. Conditions (33) after multiplication from the left and right by $\tilde{M}^{-1/2}$ become

$$KN = NK\ ,\ KG = GK\ ,\ GN = NG\ , \tag{37}$$

where $K = K^T = \tilde{M}^{-1/2}\tilde{K}\tilde{M}^{-1/2}$, $N = -N^T = \tilde{M}^{-1/2}\tilde{N}\tilde{M}^{-1/2}$, $G = -G^T = \tilde{M}^{-1/2}\tilde{G}\tilde{M}^{-1/2}$. The matrices K, N and G are real normal matrices, and, in view of (37), they pairwise commute in multiplication. Then, according to well-known result (see [4, Section 2.5]), taking into account symmetry of K and skew-symmetry of N and G, there is a single real orthogonal matrix U such that $U^T NU = \mathrm{N} = diag(\nu_1 J_2,...,\nu_m J_2,0,...,0)$ with $0 \neq \nu_j \in \Re$, $U^T KU = \Lambda = diag(\lambda_1 I_2,...,\lambda_m I_2,\lambda_{2m+1},...,\lambda_n)$ where $\lambda_j \in \Re$, and $U^T GU = \Gamma = diag\left(g_1 J_2,...,g_{[n/2]} J_2,0\right)$ with $g_j \in \Re$ ($g_j = 0$ is possible for some *j*). If $m < [n/2]$, then for any $g_{m+k} \neq 0$**,** the commutativity of Γ and Λ imposes the following additional restriction on the diagonal elements of Λ: $\lambda_{2m+2k-1} = \lambda_{2m+2k}$, $k \in \{1,...,[n/2]-m\}$. Therefore, under conditions (33), the transformation $q = \tilde{M}^{-1/2}Up$ reduces Eq. (1) to the form (20), (34)-(36). □

When $\tilde{N} = 0$, Theorem 2 coincides with a result from [8].

Example 3. Consider the system (1) with $\tilde{M} = I_4$ and

$$\tilde{K} = K = \begin{bmatrix} 3 & -1 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & -1 & 3 \end{bmatrix}, \tilde{N} = N = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \tag{38}$$

and

$$\tilde{G} = G = 2\begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}. \tag{39}$$

It is obvious that in this example the matrix $\tilde{G}$ (G) is not proportional to $\tilde{N}$ (N). This makes Theorem 1 inapplicable. On the other hand, we have

$$NG = 2\begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} = (NG)^T = GN ,$$

$$NK = \begin{bmatrix} 0 & 0 & 3 & -1 \\ 0 & 0 & -1 & 3 \\ -3 & 1 & 0 & 0 \\ 1 & -3 & 0 & 0 \end{bmatrix} = -(NK)^T = KN ,$$

and

$$GK = 8\begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} = -(GK)^T = KG ,$$

i.e., conditions (33) (or, equivalently, (37)) are satisfied, and, according to Theorem 2, the system can be transformed by a real congruence transformation into two independent two-dimensional subsystems. Indeed, the coordinate transformation $q = Up$, where

$$U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

transforms system (1),(38),(39) into the following decomposed form

$$\begin{bmatrix} \ddot{p}_1 \\ \ddot{p}_2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + 2\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = 0 ,$$

$$\begin{bmatrix} \ddot{p}_3 \\ \ddot{p}_4 \end{bmatrix} + 4\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \dot{p}_3 \\ \dot{p}_4 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} p_3 \\ p_4 \end{bmatrix} + 4\begin{bmatrix} p_3 \\ p_4 \end{bmatrix} = 0 .$$

The commutativity conditions (33), i.e. (37), impose serious restrictions on the spectrum of the matrix of potential forces. It follows from (34) that the matrix K must have multiple eigenvalues. Although this property is not generic, repeated eigenvalues of K are not so rare in the real-world applications (see [9] and the discussion therein).

Generalized coordinates q are real, and, as mentioned in the Introduction, there are no real coordinate transformations that would completely decouple (diagonalize) system (1).

However, it should be noted that conditions (33) are necessary and sufficient for complete decoupling this system by means of a *complex* *congruent transformation. Indeed, according to a result from [4], the tree normal matrices K, N and G can be simultaneously diagonalized by a unitary matrix W ($W^*W = I$, $(\ )^*$ denotes the complex conjugate transpose) if and only if conditions (37) are satisfied. Then, the complex transformation $q = \tilde{M}^{-1/2}Wz$ transforms Eq. (1) into completely decoupling form. The decoupled complex equations appear in complex conjugate pairs, and a typical such equation takes the form: $\ddot{z}_j + ig_j\dot{z}_j + (i\nu_j + \lambda_j)z_j = 0$, where λ_j, ν_j and g_j are real numbers and z_j is complex. Separating the real and the imaginary parts of the variable z_j as $z_j = \xi_j + i\eta_j$, each decoupled complex equation becomes two *coupled* ordinary differential equations in real variables ξ_j and η_j:

$$\begin{bmatrix} \ddot{\eta}_j \\ \ddot{\xi}_j \end{bmatrix} + g_j \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \dot{\eta}_j \\ \dot{\xi}_j \end{bmatrix} + \nu_j \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \eta_j \\ \xi_j \end{bmatrix} + \lambda_j \begin{bmatrix} \eta_j \\ \xi_j \end{bmatrix} = 0.$$

Therefore, in the configuration space, the completely decoupling achieved as a result of a complex transformation is only that of appearance.

In many applications in science and engineering the potential matrix of the system is positive definite ($\tilde{K} > 0$). In this case, the assumption of positive definiteness of the symmetric matrix $\tilde{M}$ can be removed. The situation where $\tilde{M}$ is positive semi-definite arises, for example, in mechanics when using redundant coordinates [10], while indefinite $\tilde{M}$ may occurs in the linear feedback control of second-order systems when the closed-loop mass matrix can be modified by acceleration feed-back [11].

If $\tilde{K} > 0$, the roles of $\tilde{M}$ and $\tilde{K}$ can be reversed in the above theorems to achieve decomposition of (1) using real congruence. More precisely, the following assertions are true.

Theorem 3. *Let* $\tilde{M} = \tilde{M}^T$, $\tilde{K} = \tilde{K}^T > 0$, $\tilde{N} = -\tilde{N}^T$, $\tilde{G} = -\tilde{G}^T$ *and* $rank\tilde{G} = 2m$. *If* $\tilde{G} = \gamma\tilde{N}$ *where* $\gamma \in \Re$, *then necessary and sufficient conditions for Eq. (1) to be transformed to equation*

$$\hat{M}\ddot{p} + \gamma\hat{N}\dot{p} + \hat{N}p + p = 0 \tag{40}$$

with

$$\hat{M} = diag(\mu_1, \mu_2, ..., \mu_n),\ \mu_j \in \Re, \tag{41}$$

and

$$\hat{N} = diag(\hat{v}_1 J_2,...,\hat{v}_m J_2,0,...,0)\ ,\ 0 \neq \hat{v}_j \in \Re\ , \tag{42}$$

using a real linear change of coordinates are

$$\tilde{M}\tilde{K}^{-1}\tilde{N}\tilde{K}^{-1}\tilde{N} = \tilde{N}\tilde{K}^{-1}\tilde{N}\tilde{K}^{-1}\tilde{M} \tag{43}$$

and

$$(\tilde{M}\tilde{K}^{-1}\tilde{N}\tilde{K}^{-1})^2 = (\tilde{N}\tilde{K}^{-1}\tilde{M}\tilde{K}^{-1})^2\ . \tag{44}$$

The real numbers μ_j (j = 1,…,n) in (41) are eigenvalues of the matrix $\tilde{K}^{-1}\tilde{M}$, while the numbers $\pm i\,\hat{v}_j$ (j = 1,…,m) in (42) are nonzero eigenvalues of $\tilde{K}^{-1}\tilde{N}$.

Theorem 4. *Let* $\tilde{M} = \tilde{M}^T$, $\tilde{K} = \tilde{K}^T > 0$, $\tilde{N} = -\tilde{N}^T$, $\tilde{G} = -\tilde{G}^T$ *and* $rank\tilde{N} = 2m$. *If the following tree conditions are satisfied*

$$\tilde{M}\tilde{K}^{-1}\tilde{N} = \tilde{N}\tilde{K}^{-1}\tilde{M}\ ,\ \tilde{M}\tilde{K}^{-1}\tilde{G} = \tilde{G}\tilde{K}^{-1}\tilde{M}\ ,\ \tilde{G}\tilde{K}^{-1}\tilde{N} = \tilde{N}\tilde{K}^{-1}\tilde{G}\ , \tag{45}$$

then there exists a real linear change of coordinates that transforms Eq. (1) to the form

$$\hat{M}\ddot{p} + \hat{\Gamma}\dot{p} + \hat{N}p + p = 0$$

with

$$\hat{M} = diag(\mu_1 I_2,...,\mu_m I_2,\mu_{2m+1},...,\mu_n)\ ,\ \hat{N} = diag(\hat{v}_1 J_2,...,\hat{v}_m J_2,0,...,0)\ ,\ \mu_j \in \Re\ ,0 \neq \hat{v}_j \in \Re\ ,$$

and

$$\hat{\Gamma} = diag\left(\hat{g}_1 J_2,...,\hat{g}_{[n/2]} J_2,0\right),\ g_j \in \Re\ .$$

If $m < [n/2]$, *then for any* $\hat{g}_{m+k} \neq 0$, $\mu_{2m+2k-1} = \mu_{2m+2k}$, $k \in \{1,...,[n/2]-m\}$ **.**

Remark 6. If $\tilde{M} = \tilde{M}^T > 0$ and $\tilde{K} = \tilde{K}^T > 0$, then Theorem 3 and Theorem 4 are equivalent to Theorem 1 and Theorem 2, respectively. Also, if the potential matrix $\tilde{K}$ is negative definite Theorem 3 and Theorem 4 can be applied after premultiplication Eq. (1) by -1.

Remark 7. When $\tilde{K} > 0$, conditions (45) are necessary and sufficient for complete decomposition (diagonalization) of the system using a complex *congruent transformation.

4. CONCLUSION

This paper deals with linear *n*-degree-of-freedom systems with potential, circulatory and gyroscopic forces. Since circulatory and gyroscopic matrices are skew, it precludes the decomposition of such a system into *n* uncoupled systems through the use of a *real* coordinate change. The best that can be done using a real coordinate change is to decouple the system into subsystems each of which has at most two-degrees-of-freedom. The conditions for such a decoupling are provided here. Below, we summarize the main findings.

When the coefficient matrix of gyroscopic forces is proportional to the circulatory one then an *n*-degree-of-freedom linear circulatory gyroscopic potential system in which the circulatory matrix has rank $2m \leq n$ can be decomposed by a suitable real linear change of coordinates into *m* uncoupled two-degree-of-freedom subsystems and $(n-2m)$ single-degree of freedom subsystems if and only if the two conditions obtained in the paper are satisfied. The two-degree-of-freedom subsystems are each circulatory gyroscopic potential subsystems, while the single-degree-of-freedom subsystems are each pure potential systems. If the circulatory matrix has distinct non-zero eigenvalues then the two necessary and sufficient conditions for decoupling the dynamical system reduce to just a single necessary and sufficient condition.

If the potential, circulatory and gyroscopic matrices pairwise commute with respect to the inverse of mass matrix, then the dynamical system decouples through the use of a suitable real congruence into a series of independent two-degree-of-freedom subsystems in canonical form and single-degree-of-freedom ones. The potential matrix of each two-degree-of-freedom subsystem with circulatory and/or gyroscopic terms is proportional to the identity matrix.

The analogous results are also formulated for a circulatory gyroscopic potential system when the potential matrix of the linear dynamical system is assumed to be positive definite and the mass matrix is assumed to be only symmetric.

Several illustrative examples are considered throughout the paper to give clarity of the analytical results that are obtained.

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O KONGRUENTNOM RASPREZANJU CIRKULACIONIH GIROSKOPSKIH SISTEMA

Sažetak:

Razmatra se mogućnost rasprezanja linearnih dinamičkih sistema sa potencijalnim, cirkulacionim i giroskopskim silama. Dokazani su kriterijumi koji sadrže uslove egzistencije realnih linearnih koordinatnih transformacija, koje dovode do razdvajanja ovih sistema sa konačnim brojem stepena slobode na nezavisne podsisteme stepena slobode ne većeg od dva. Iz ovih kriterijuma, kao posljedice, dobijeno je više specifičnih rezultata. Rezultati su ilustrovani sa nekoliko numeričkih primjera.

Ključne riječi: linearni sistem, potencijalne sile, cirkulacione sile, giroskopske sile, rasprezanje, kongruentna transformacija